Topological Quantum Computing with Read-Rezayi States

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Read-Rezayi fractional quantum Hall states are among the prime candidates for realizing non-Abelian anyons which in principle can be used for topological quantum computation. We present a prescription for efficiently finding braids which can be used to carry out a universal set of quantum gates on encoded qubits based on anyons of the Read-Rezayi states with $k > 2$, $k \neq 4$. This work extends previous results which only applied to the case $k = 3$ (Fibonacci) and clarifies why in that case gate constructions are simpler than for a generic Read-Rezayi state.

Non-Abelian anyons [1] — quasiparticle excitations obeying so-called non-Abelian statistics — are conjectured to emerge in certain two-dimensional quantum systems. In these systems, when well-separated anyons are present, there is a ground state degeneracy that grows exponentially with the number of anyons. Furthermore, if these anyons remain well-separated, different states in the ground state manifold cannot be distinguished by local measurements — thus rendering this space immune from decoherence due to any local perturbations.

When non-Abelian anyons are exchanged, the process is described by a multidimensional unitary operation (instead of a single phase) acting on the degenerate space. Certain unitary operations can then be carried out by dragging anyons around one another, "braiding" their worldlines in 2+1 dimensional space-time. As long as the anyons are kept sufficiently far apart during this process, the resulting unitary operation will be identical for any two topologically equivalent braids.

Recent interest in non-Abelian anyons has focused on the possibility of using them for topological quantum computation — a form of quantum computation which exploits the topological robustness of braiding and the protection of the degenerate Hilbert space against decoherence to process and store quantum information in an intrinsically fault-tolerant way [2, 3, 4]. This paper is concerned with the problem of finding specific braiding patterns which can be used to carry out universal quantum computation for a class of non-Abelian anyons described by $\mathfrak{su}(2)_k$ Chern-Simons-Witten theories.

The theory of $\mathfrak{su}(2)_k$ anyons provides the mathematical description [5] (up to Abelian phases not relevant here) of the braiding properties of quasiparticle excitations in the Read-Rezayi [6] sequence of fractional quantum Hall (FQH) states. Here the parameter k (the "level") is a positive integer characterizing the state. For example, the $k = 2$ state is the Moore-Read state [1], believed to describe the FQH plateau observed at filling fraction $\nu = 5/2$ [4], and the $k = 3$ state may describe the FQH plateau observed at $\nu = 12/5$ [6]. Bosonic Read-Rezayi states may also be realizable in rotating Bose gases [7], and model spin Hamiltonians have been constructed [8]

for which the low-energy quasiparticle excitations can be described by any consistent achiral theory of non-Abelian anyons, including (doubled) $\mathfrak{su}(2)_k$, suggesting the possibility of realizing $\mathfrak{su}(2)_k$ anyons in exotic spin liquids.

It has been shown that $\mathfrak{su}(2)_k$ anyons with $k = 3$ or $k > 4$ can be used to carry out universal quantum computation just by braiding anyons [3]. In previous work [9, 10] a number of different prescriptions have been given for explicitly constructing braids to carry out universal quantum computation using $\mathfrak{su}(2)_3$ anyons — anyons which are, for our purposes, essentially equivalent to the so-called Fibonacci anyons [4]. In the present work, we extend these results to all $\mathfrak{su}(2)_k$ anyons with $k \geq 3$, $k \neq 4$.

$\mathfrak{su}(2)_k$ anyons carry a quantum number resembling ordinary spin referred to here as (topological) "charge" [4, 11]. For the level k theory the allowed values of this charge include all integers and half integers between 0 and $k/2$. Similar to ordinary spin, there are rules for combining topological charge which specify the possible total charge of objects formed when two or more anyons are combined. For $\mathfrak{su}(2)_k$ anyons the fusion rule is a truncated version of the usual triangle rule for adding angular momenta,

$$s_1 \otimes s_2 = |s_1 - s_2| \oplus (|s_1 - s_2| + 1) \oplus \cdots \quad (1)$$
$$\oplus \ \min(s_1 + s_2, k - (s_1 + s_2)).$$

From this fusion rule it can be shown that, asymptotically, the dimensionality of the Hilbert space of N identical anyons grows as d_k^N where d_k is known as the quantum dimension of the particles [4, 11]. For charge $1/2$ anyons $d_k = \lfloor 2 \rfloor_q$, where the q-integer $\lfloor m \rfloor_q$ is defined as $\lfloor m \rfloor_q = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$ and $q = e^{i2\pi/(k+2)}$. For example, in the case of ordinary spin $1/2$ particles (corresponding to $\mathfrak{su}(2)_k$ anyons when $k \to \infty$) the quantum dimension is 2, as expected, while for Fibonacci anyons ($k = 3$), where the dimensionality of the Hilbert space grows as the Fibonacci sequence, the quantum dimension is the golden ratio.

For all finite $k \geq 2$, the quantum dimension is an irrational number between 1 and 2. It follows that the Hilbert space of N charge $1/2$, $\mathfrak{su}(2)_k$ anyons cannot be
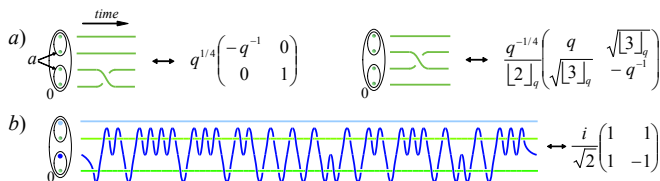
FIG. 1: (a) Encoded qubits and elementary braid matrices for $\mathfrak{su}(2)_k$ anyons, and (b) a sample single-qubit gate. Here, and in subsequent figures, groups of anyons in charge eigenstates are enclosed in ovals labeled by the charge. Qubits are encoded in the two-dimensional Hilbert space of four charge $1/2$ anyons with total charge 0. The matrices shown, in the basis labeled by $a = \{0, 1\}$, correspond to the elementary braid operations, also shown. Within the encoded qubit space, braiding the top two anyons produces the same unitary operation as braiding the bottom two. The braid depicted in (b) consists of 80 interchanges and approximates a Hadamard gate for $\mathfrak{su}(2)_5$ anyons with an accuracy of $\sim 2.2 \times 10^{-6}$ (measured using operator norm, see [9]).

decomposed into a tensor product of smaller subsystems. To carry out quantum computation within the standard "qubit plus quantum gate" framework one must therefore encode qubits using several anyons. Here we encode qubits in the two-dimensional Hilbert space of four charge $1/2$ anyons with total charge 0 [12]. Referring to Fig. 1, we choose the logical $|0\rangle$ and $|1\rangle$ for this qubit to be states for which the total charge of the two bottommost (or, equivalently, the two topmost) anyons is 0 and 1, respectively. Note that the fusion rule (1) indicates that when the constraint that the total charge be 0 is relaxed the Hilbert space of four charge $1/2$ anyons is five-dimensional for $k = 3$ and six-dimensional for $k > 3$. The states that do not correspond to qubit states, i.e. states with total charge 1 or 2, are then *non-computational states*. The existence of these states implies the possibility of making transitions from the qubit space to the non-computational space, i.e. leakage errors, which pose a major problem in constructing quantum gates.

To carry out universal quantum computation it must be possible to perform arbitrary single-qubit gates as well as at least one entangling two-qubit gate. It has been shown that for $\mathfrak{su}(2)_k$ anyons when $k \geq 3$, $k \neq 4, 8$, any desired single-qubit gate can be approximated, to any desired accuracy, by weaving a charge $1/2$ anyon around two others within a qubit [3, 13]. In practice, braids which approximate a given single-qubit gate can be found by carrying out a search over braids up to a given length and choosing those which produce unitary operations closest to the desired target gate. As an example, Fig. 1 shows a braid which is the result of a bidirectional search [14] and which approximates a Hadamard gate using $\mathfrak{su}(2)_5$ anyons to an accuracy of 1 part in 10^6. Given the ability to perform a search this deep, further accuracy can always be systematically achieved by ap-

plying the Solovay-Kitaev algorithm [9]. Note that for single-qubit gates there is no danger of leakage errors because braiding within a qubit cannot change its total topological charge.

When searching for braids we are performing a discrete search over a continuous space. The dimensionality of the space being searched, D, is the main factor which determines the efficacy of these searches. If the number of distinct braids one can search is N_b, the typical error for approximating a given target gate will be $\sim N_b^{-1/D}$. Thus, for fixed N_b, increasing D greatly reduces the accuracy of the braids one can obtain. For single-qubit gates the search space is $SU(2)$ with $D = 3$ which is sufficiently small to allow us to find highly accurate braids, as demonstrated by the above example.

Two-qubit gates are significantly harder to construct, both due to the possibility of leakage errors, and the fact that for eight anyons the search space is $SU(13)$ for $k = 3$ and $SU(14)$ for $k > 3$ (with dimensionalities of 168 and 195, respectively). The key idea for efficiently finding braids for two-qubit gates, introduced in [9] and common to more recent approaches [10], is to reduce the two-qubit gate problem to one or more effective "single-qubit" problems, i.e. problems in which one is searching over braids which involve only three objects at time (where an object can be either a single anyon, or a collection of anyons braided as a single entity), and for which the search space is $SU(2)$. In this case, as emphasized above, excellent approximations for any desired target operation can be obtained.

In addition to reducing all searches to $SU(2)$, previous work has focused on the case $k = 3$. To see why $k = 3$ is special note that in this case, for anyons with topological charge 1, the only non-trivial fusion rule is $1 \otimes 1 = 0 \oplus 1$, which is the fusion rule of the Fibonacci anyons [4]. All previous gate constructions have exploited in one way or another the unique feature of Fibonacci anyons that there is only one nontrivial value of the topological charge. Thus any collection of Fibonacci anyons either has topological charge 0 and is therefore "neutral" (it does not induce any non-Abelian transitions if braided as a single cohesive object) or has topological charge 1, in which case it behaves as a single Fibonacci anyon.

We illustrate the usefulness of these features with a simple two-qubit gate construction [15] (see also [10]). Figure 2(a) shows a two-qubit braid in which a pair of anyons from the control qubit (the control pair) is woven around *pairs* of anyons in the target qubit, before returning to its original position. When the control qubit is in the state $|0\rangle$ the control pair has charge 0 and, because weaving a charge 0 object around other anyons does not induce any non-Abelian transitions, the result of this operation will be trivial (i.e. the identity). Similarly, when the target qubit is in the state $|0\rangle$, the control pair is woven around objects with total charge 0 and the result will again be the identity (regardless of the state of the con-
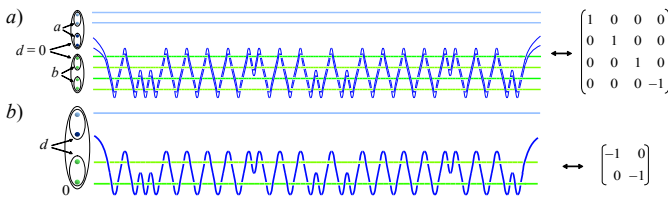
FIG. 2: (Color Online). "Effective qubit" gate construction for $\mathfrak{su}(2)_3$ anyons. Part (a) shows a braid in which a pair of anyons from the control qubit (shown in blue) weaves around pairs of anyons in the target qubit (shown in green). When either the control or target qubits are in the state $|0\rangle$ this braid produces the identity operation. When both control and target qubits are in the state $|1\rangle$ the braid consists of weaving a charge 1 anyon around two other charge 1 anyons, as shown in (b). For $\mathfrak{su}(2)_3$ anyons the braiding properties of charge $1/2$ anyons and charge 1 anyons are the same (up to irrelevant Abelian phases), and the problem is reduced to that of finding a particular single-qubit gate by carrying out a search in $SU(2)$. The braid shown in (b) approximates a negative identity matrix, in the basis labeled by $d = \{0, 1\}$, for $\mathfrak{su}(2)_3$ anyons with an accuracy of $\sim 1.6 \times 10^{-5}$. As a consequence, the full braid shown in (a) approximates a controlled-Z gate, shown in the basis labeled by $ab = \{00, 01, 10, 11\}$, with an accuracy of $\sim 1.4 \times 10^{-5}$.

trol qubit). Thus, by construction this weave acts as the identity on the two-qubit states $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, and $|1\rangle|0\rangle$, with the only nontrivial case being $|1\rangle|1\rangle$ (here the first qubit is the control qubit). To construct an entangling two-qubit gate it is then necessary to find a particular weave of the form shown in Fig. 2(a) which returns the state $|1\rangle|1\rangle$ to itself while acquiring a nontrivial phase with respect to the state $|1\rangle|0\rangle$, thus producing a controlled rotation of the target qubit.

Finding such a weave is straightforward for Fibonacci anyons $(k = 3)$. In this case the fusion rule implies that the Hilbert space of four charge 1 objects with total charge 0 is two-dimensional and the problem reduces to that of searching for a particular single-qubit operation acting on the "effective qubit" shown in Fig. 2(b). The states of this "qubit" are determined by the label d which can be either 0 or 1. To ensure there are no transitions between the encoded qubit states $(d = 0)$ and the non-computational states $(d = 1)$, the resulting unitary operation must be diagonal in d. As an example, Fig. 2(b) shows a braid which approximates a negative identity matrix. If one follows this braiding pattern by weaving the control pair around pairs of anyons in the target qubit (as shown in Fig. 2(a)), the $|1\rangle|1\rangle$ state acquires a phase of -1 and the resulting two-qubit gate is a controlled-Z gate which is equivalent to a CNOT, up to single-qubit rotations.

We now turn to the case $k > 3$. In this case the fusion rule for combining two charge 1 objects is $1 \otimes 1 = 0 \oplus 1 \oplus 2$. This implies that d, the overall charge of the original

qubits shown in Fig. 2, can now take three different values. Hence, when $k > 3$ the unitary operations corresponding to braids of the form shown in Fig. 2(a) are elements of $SU(3)$, not $SU(2)$. It is in principle possible to carry out a search in $SU(3)$, but because the dimensionality of the search space is 8, rather than 3, it is significantly less efficient than searching $SU(2)$.

In general we find that for $k > 3$ it is impossible to construct a leakage free two-qubit entangling gate by performing a single braid in which three objects are braided and the search space is $SU(2)$ [16]. However, it *is* possible to construct such a gate by breaking the construction into three steps, as illustrated in Fig. 3. In each of these steps a pair of anyons from the control qubit (again, the control pair) is woven around two anyons in the target qubit.

Before describing the details of the construction we establish the key fact that finding braids for each step only requires a search in $SU(2)$. As before we need only consider the case when the control pair has charge 1 $(a = 1)$. Each step then involves weaving a charge 1 object around two charge $1/2$ anyons. According to the fusion rule (1) the Hilbert space of these three objects decouples into two one-dimensional sectors (with total charge 0 and 2) and a single two-dimensional sector (with total charge 1). The action on the one-dimensional sectors is determined entirely by the winding number of the braid — in particular if we fix the winding number to be 0 the action is trivially the identity [9]. The only nontrivial action is then on the two-dimensional sector, for which the relevant search space is $SU(2)$. For $\mathfrak{su}(2)_k$ anyons it is straightforward to determine the relevant braid matrices [11, 15] and we find that for all $k \geq 3$, $k \neq 4$ these matrices generate dense covers of $SU(2)$, (this includes the case $k = 8$ for which braiding three charge $1/2$ objects is *not* dense in $SU(2)$ [3]). We therefore expect, and do indeed find, that for each step in our construction carrying out a search can produce braids which approximate the desired operations to an accuracy of few parts in 10^6.

Now we turn to the actual construction. The first step consists of a braid which effectively "swaps" the control pair with a pair of anyons in the target qubit. Figure 3(a) shows this step for $\mathfrak{su}(2)_5$ anyons. Referring to this figure, the control pair starts from the bottom position in the control qubit, weaves around the two topmost anyons in the target qubit, in the end swapping positions with them. The specific braid shown is the result of a search for a braid which generates a unitary operation approximating the matrix shown in the figure. (In this matrix — and the one shown for step two — the upper left and lower right elements correspond to the one-dimensional sectors with total charge 0 and 2, and the middle 2×2 block acts on the two-dimensional sector with total charge 1, described above). This operation is designed to effectively swap the control pair with the topmost pair of anyons in the target qubit (labeled b),
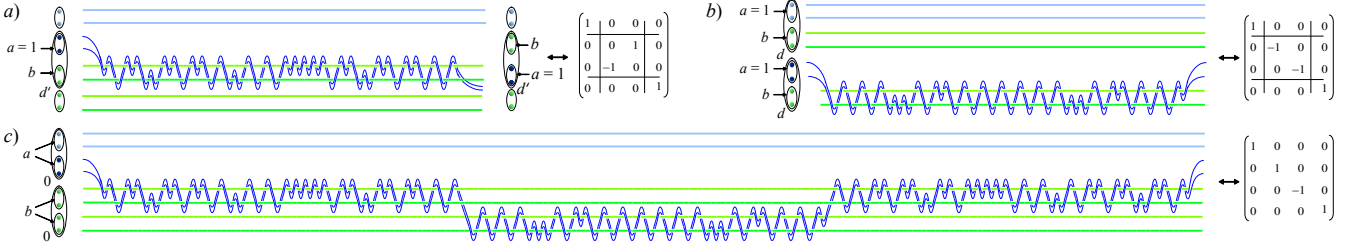
FIG. 3: (Color Online) Controlled-Phase gate construction for $\mathfrak{su}(2)_5$ anyons. Part (a) shows a swap braid which effectively exchanges the control pair (shown in blue) with the topmost pair of anyons in the target qubit (green). This braid approximates the unitary operation shown in the figure with an accuracy of $\sim 5.1 \times 10^{-6}$. Note that the matrix shown is expressed using inequivalent left and right bases states labeled $bd' = \{10, 01, 11, 12\}$ at both the start and end of the braid. Part (b) shows a braid which acts on the intermediate state produced by the swap braid and gives the state with $a = 1$ $b = 0$ a phase of -1. The corresponding unitary operation shown in the figure in the basis labeled by $bd = \{10, 01, 11, 12\}$ is approximated with an accuracy of $\sim 7.0 \times 10^{-6}$. (c) Full controlled-$(-Z)$ gate, in the basis labeled by $ab = \{00, 01, 10, 11\}$, with an accuracy of $\sim 7.4 \times 10^{-7}$.

and when $a = b = 1$ it does so without disturbing the quantum numbers of the system. This means that if the initial state of the qubit is $|1\rangle|1\rangle$ the final state will be the same but with the control pair now swapped into the target.

The net effect of this swap operation is to take the system to the intermediate state shown as the starting state in Fig. 3(b). In this intermediate state all the information of the initial state of the two qubits is encoded in the bottom four anyons. In particular (assuming $a = 1$) the total charge of these anyons, labeled d in the figure, must be 0 if $b = 1$ (since in this case the swap operation is essentially the identity) and 1 if $b = 0$ (due to the fusion rules).

To perform an entangling two-qubit gate we need only induce a phase shift between the $b = 0$, and $b = 1$ ($d = 0$) states when $a = 1$. This is done in the second step of our construction, in which the control pair is woven around the bottom two anyons and returned to its starting position (see Fig. 3(b)). The weave shown is the result of a search which produces a unitary operation that is diagonal in b and gives the state $a = 1$, $b = 0$ a nontrivial phase (-1 for the braid shown).

In the third and final step the control pair is returned to its original position in the control qubit by applying the inverse of the swap braid. Putting all three steps together, the resulting full braid is shown in Fig. 3(c). If the control qubit is in the state $|0\rangle$, the control pair has charge 0 and the effect of this braid is simply the identity operation. If the control qubit is in the state $|1\rangle$, this braid first swaps the control pair into the target, then, if the target qubit was initially in the state $|0\rangle$, gives the resulting intermediate state a phase $e^{i\phi}$, and finally returns the control pair to the control qubit. The full braid then approximates a controlled-phase gate for which, if the control qubit is in the state $|1\rangle$, the target qubit is rotated about the $\hat{z}$ axis by the angle ϕ. When $\phi = \pi$, as is the case for the braid shown in the figure, this gate is a controlled-$(-Z)$ gate which is equivalent to a CNOT, up to single-qubit rotations.

We conclude by pointing out that for certain values of k it is possible to carry out step two of our construction with a finite braid in such a way that the phase difference between the state $a = 1$, $b = 0$ and $a = 1$, $b = 1$ ($d = 0$) is *exactly* -1. Specifically, for $k = 8n - 2$ this can be done by weaving the control pair completely around the two bottom anyons in the target qubit n times.

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[16] One might consider constructing a two-qubit gate by weaving the control pair around two anyons in the target qubit, or weaving it around a pair and a single anyon. It can be shown that the former cannot lead to an entangling two-qubit gate when $k > 3$ [9] while the latter requires a search in $SU(2) \oplus SU(2)$ with a six-dimensional search space.